

ON MAXIMAL TAIL PROBABILITY OF SUMS OF NONNEGATIVE, INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES

TOMASZ ŁUCZAK, KATARZYNA MIECZKOWSKA, AND MATAS ŠILEIKIS

ABSTRACT. We consider the problem of finding the optimal upper bound for the tail probability of a sum of k nonnegative, independent and identically distributed random variables with given mean x . For $k = 1$ the answer is given by Markov's inequality and for $k = 2$ the solution was found by Hoeffding and Shrikhande in 1955. We solve the problem for $k = 3$ as well as for general k and $x \leq 1/(2k - 1)$ by showing that it follows from the fractional version of an extremal graph theory problem of Erdős on matchings in hypergraphs.

1. INTRODUCTION

The purpose of this paper is to consider the problem of finding, for $x, t \geq 0$, the quantity

$$(1.1) \quad \sup_{\mathbf{X}} \mathbb{P}(X_1 + \dots + X_k \geq t),$$

where the supremum is taken over all random vectors $\mathbf{X} = (X_1, \dots, X_k)$ of nonnegative, independent and identically distributed (further *i.i.d.*) random variables X_i such that $\mathbb{E}(X_i) \leq x$ for $i = 1, \dots, k$.

From now on we assume that $t = 1$, since, writing

$$m_k(x) = \sup_{\mathbf{X}} \mathbb{P}(X_1 + \dots + X_k \geq 1),$$

by rescaling we get that (1.1) is equal to $m_k(x/t)$.

For $x \geq 1/k$ the trivial solution $m_k(x) = 1$ is given by X_i 's which are identically equal to x . For $k = 1$ and $x < 1$ the solution $m_1(x) = x$ is given by Markov's inequality and a zero-one random variable. In the case of two variables the problem was solved by Hoeffding and Shrikhande [6] who showed that

Date: December 14, 2015.

The first author is partially supported by NCN grant 2012/06/A/ST1/00261.

The second author is partially supported by NCN grant 2012/05/N/ST1/02773.

$$m_2(x) = \begin{cases} 2x - x^2 & \text{for } x < 2/5; \\ 4x^2 & \text{for } 2/5 \leq x < 1/2; \\ 1, & \text{for } x \geq 1/2. \end{cases}$$

We conjecture that the following generalization of the above results holds.

Conjecture 1.1. *For every positive integer k and $x \geq 0$ we have*

$$(1.2) \quad m_k(x) = \begin{cases} 1 - (1 - x)^k & \text{for } x < x_0, \\ (kx)^k & \text{for } x_0 \leq x < 1/k, \\ 1, & \text{for } x \geq 1/k, \end{cases}$$

where $x_0(k)$ is the solution of $1 - (1 - x)^k = (kx)^k$.

Note that the lower bound on $m_k(x)$ in the first case is given by X_i 's with a two-point distribution $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = 0) = x$ and in the second case by X_i 's with distribution $\mathbb{P}(X_i = 1/k) = 1 - \mathbb{P}(X_i = 0) = kx$.

The case when X_i 's are not necessarily identically distributed has also been studied. Let

$$s_k(x) = \sup_{\mathbf{X}} \mathbb{P}(X_1 + \cdots + X_k \geq 1),$$

where the supremum is taken over all vectors of independent, nonnegative random variables with common mean x . Clearly, $m_k(x) \leq s_k(x)$. In 1966 Samuels [8] formulated a conjecture on the least upper bound for the tail probability in terms of $\mathbb{E}X_i, i = 1, \dots, k$, which are not necessarily equal. For simplicity, we state this conjecture in the case when means are equal.

Conjecture 1.2 (Samuels [8]). *For every positive integer k and $x \geq 0$ we have*

$$(1.3) \quad s_k(x) = \begin{cases} 1 - \min_{t=0}^{k-1} \left(1 - \frac{x}{1-tx}\right)^{k-t} & \text{for } x < 1/k, \\ 1, & \text{for } x \geq 1/k, \end{cases}$$

The lower bound on $s_k(x)$ is given by one of the random vectors $\mathbf{X}_t, t = 0, \dots, k-1$, where \mathbf{X}_t consists of t random variables identically equal to x and $k-t$ i.i.d. random variables taking values 0 and $1-tx$. Therefore, if true, (1.3) implies Conjecture 1.1 only when the minimum is attained by $t = 0$. Computer-generated graphs suggest that the minimum is attained by $t = 0$ when $x < x_1(k)$ and by $t = k-1$ when $x \geq x_1(k)$, where $x_1(k) \in (0, 1/k)$ is the solution of $1 - (1 - x)^k =$

$x/(1 - (k - 1)x)$. In [1] it was shown rigorously that the minimum is attained by $t = 0$ for $x \leq 1/(k + 1)$.

Samuels [8, 9] confirmed (1.3) for $k = 3, 4$. Computer-generated graphs of functions $s_3(x)$ and $s_4(x)$ suggest that for $k = 3, 4$ we have

$$m_k(x) = s_k(x) = 1 - (1 - x)^k, \quad x \leq x_1(k),$$

where $x_1(3) = 0.27729\dots$, $x_1(4) = 0.21737\dots$.

Moreover, Samuels [10] proved that for $k \geq 5$

$$m_k(x) = s_k(x) = 1 - (1 - x)^k, \quad x \leq 1/(k^2 - k).$$

Our main result can be stated as follows.

Theorem 1.3. *Conjecture 1.1 holds for $k = 3$ and every x . Moreover, it holds for $k \geq 5$ when $x < \frac{1}{2k-1}$.*

The proof of Theorem 1.3 is based on an observation that Conjecture 1.1 is asymptotically equivalent to the fractional version of Erdős' Conjecture on matchings in hypergraphs which we introduce in the next section.

2. THE HYPERGRAPH MATCHING PROBLEM

A k -uniform hypergraph $H = (V, E)$ is a set of vertices V together with a family E of k -element subsets of V , called *edges*. A *matching* is a family of disjoint edges of H , and the size of the largest matching in H is called a *matching number* and is denoted by $\nu(H)$.

In [2] Erdős stated the following.

Conjecture 2.1 (Erdős [2]). *Let $H = (V, E)$ be a k -uniform hypergraph, $|V| = n$, $\nu(H) = s$. If $n \geq ks + k - 1$, then*

$$(2.1) \quad |E| \leq \max \left\{ \binom{n}{k} - \binom{n-s}{k}, \binom{ks+k-1}{k} \right\}.$$

Note that the equality in Conjecture 2.1 holds either when H is a hypergraph consisting of all k -element sets intersecting a given subset $S \subset V$, $|S| = s$, or when H consists of all k -element subsets of a given subset $T \subset V$, $|T| = ks + k - 1$. We denote these two families of hypergraphs by $Cov_{n,k}(s)$ and $Cl_{n,k}(ks + k - 1)$, respectively.

A similar problem can be formulated in terms of fractional matchings. A *fractional matching* in a hypergraph H is a function

$w : E \rightarrow [0, 1]$ such that

$$\sum_{e \ni v} w(e) \leq 1 \text{ for every vertex } v \in V.$$

Then, $\sum_{e \in E} w(e)$ is a *size* of the matching w and a size of the largest fractional matching in H , denoted by $\nu^*(H)$, is a *fractional matching number*. Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1] stated the following conjecture.

Conjecture 2.2 ([1]). *Let $x \in [0, 1/k]$ be fixed and let $H_n = (V_n, E_n)$ be a sequence of k -uniform hypergraphs such that $\nu^*(H_n) \leq x|V_n|$. Then*

$$(2.2) \quad \limsup_{n \rightarrow \infty} \frac{|E_n|}{\binom{|V_n|}{k}} \leq \max \{1 - (1 - x)^k, (kx)^k\}.$$

Finding a fractional matching number is a linear programming problem. Its dual problem is to minimize the size of a *fractional vertex cover* of H , which is defined as a function

$$\begin{aligned} w : V &\rightarrow [0, 1] \text{ such that} \\ \text{for each } e \in E &\text{ we have } \sum_{v \in e} w(v) \geq 1. \end{aligned}$$

Then, $\sum_{v \in V} w(v)$ is the *size* of w and the size of the smallest fractional vertex cover in H is denoted by $\tau^*(H)$. By the Duality Theorem,

$$(2.3) \quad \nu^*(H) = \tau^*(H).$$

The bound in Conjecture 2.2, if true, is attained by either a sequence $H_n \in \text{Cov}_{n,k}(\lfloor xn \rfloor)$ (which has a fractional vertex cover $w(v) = \mathbf{1}_{v \in S}$ of size $\lfloor xn \rfloor$ and therefore satisfies $\nu^*(H_n) \leq xn$) or $H_n \in \text{Cl}_{n,k}(\lfloor kxn \rfloor)$ (which has fractional vertex cover $w(v) = \frac{1}{k} \mathbf{1}_{v \in T}$ of size $\lfloor kxn \rfloor / k$).

Observe that if a fractional matching w is such that $w(e) \in \{0, 1\}$ for every edge e , then w is just a matching or, more precisely, the indicator function of a matching. Thus, every integral matching is also a fractional matching and hence

$$(2.4) \quad \nu(H) \leq \nu^*(H),$$

so consequently, Conjecture 2.2 follows from Conjecture 2.1. Furthermore, Conjecture 2.1 was confirmed for $k = 3$ by the first two authors [7] (for n bigger than some absolute constant) and by Frankl [4] (for every n). Moreover, Frankl [5] confirmed Conjecture 2.1 for $k \geq 4$ and $s \leq (n - k)/(2k - 1)$. Therefore, in view of (2.4) we have the following.

Remark 2.3 ([4, 5, 7]). Conjecture 2.2 holds for $k = 3$ and every x as well as for $k \geq 4$ and $x < 1/(2k - 1)$.

3. PROOF OF THEOREM 1.3

We prove Theorem 1.3 in two steps. First we observe that it is enough to confirm Conjecture 1.1 with some additional restrictions. Then we show the equivalence of Conjectures 1.1 and 2.2.

Here and below, given a discrete random variable, $\text{supp}(X)$ denotes the set of values which X attains with positive probability.

Lemma 3.1. *It suffices to prove Conjecture 1.1 for X_i 's with discrete distribution satisfying the following properties: (i) $\text{supp}(X_i)$ is finite subset of $[0, 1]$; (ii) $\mathbb{P}(X_i = a) \in \mathbb{Q}$ for every $a \in \text{supp}(X_i)$.*

Proof. Define, for $x \geq 0$,

$$M(x) = \max\{1 - (1 - x)^k, (kx)^k, 1\},$$

which is equal to the right-hand side of (1.2).

Let us first assume that Conjecture 1.1 holds for random variables satisfying (i) and (ii) and show it holds for X_i 's satisfying (i) only, that is, with $\text{supp}(X_i) = \{a_1, \dots, a_m\} \subset [0, 1]$. Let $p_j = \mathbb{P}(X_i = a_j)$, $j = 1, \dots, m$. For every sufficiently large integer n define a random variable $Y_i^{(n)}$ such that

$$\mathbb{P}(Y_i^{(n)} = a_j) = p_j^{(n)} \in \mathbb{Q}, \quad j = 1, \dots, m$$

where $p_j^{(n)} = \lceil np_j \rceil / n$ for $j = 2, \dots, m$, and $p_1^{(n)} = 1 - \sum_{j=2}^m p_j^{(n)}$ (note that the $p_1^{(n)}$ is positive for n large enough). Then, for every j we have $p_j^{(n)} \leq p_j + 1/n$, and therefore

$$\mathbb{E}(Y_i^{(n)}) \leq \mathbb{E}(X_i) + m/n.$$

Applying the conclusion of Conjecture 1 to $Y_i^{(n)}$'s and using the continuity of function M , we get

$$\mathbb{P}\left(\sum_{i=1}^k X_i \geq 1\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^k Y_i^{(n)} \geq 1\right) \leq \lim_{n \rightarrow \infty} M(x + m/n) = M(x).$$

Let us now assume that the Conjecture 1.1 holds for random variables satisfying (i) and show it then holds for arbitrary $X_i, i = 1, \dots, k$. For every positive integer m define random variables

$$Y_i^{(m)} = \min\{\lceil mX_i \rceil / m, 1\}.$$

Note that $\text{supp}(Y_i^{(m)})$ is finite and contained in $[0, 1]$. We have $Y_i^{(m)} \leq X_i + 1/m$ and therefore $\mathbb{E}(Y_i^{(m)}) \leq \mathbb{E}(X_i) + 1/m \leq x + 1/m$. For sufficiently large m we get

$$\mathbb{P}\left(\sum_{i=1}^k X_i \geq 1\right) \leq \mathbb{P}\left(\sum_{i=1}^k \lceil mX_i \rceil / m \geq 1\right) = \mathbb{P}\left(\sum_{i=1}^k Y_i^{(m)} \geq 1\right) \leq M(x + 1/m).$$

Taking a limit over $m \rightarrow \infty$ and using the fact that M is continuous (from the right), we obtain

$$\mathbb{P}\left(\sum_{i=1}^k X_i \geq 1\right) \leq M(x). \quad \square$$

Lemma 3.2. *For every k and $x \in [0, 1/k]$ Conjectures 1.1 and 2.2 are equivalent.*

Proof. The proof that Conjecture 1.1 implies Conjecture 2.2 goes along the same lines as the proof of Theorem 2.1 in [1]. We recall it below for the sake of completeness.

Let us fix k and $x \in [0, 1/k]$ and suppose that Conjecture 1.1 holds. Moreover, let $H_n = (V_n, E_n)$ be a sequence of k -uniform hypergraphs such that $\nu^*(H_n) \leq x|V_n| = xn$. By (2.3) we have $\tau^*(H_n) = \nu^*(H_n) \leq xn$, hence there exists a weight function $w_n : V_n \rightarrow [0, 1]$ such that

$$\sum_{v \in V_n} w_n(v) = xn,$$

and $\sum_{v \in e} w_n(v) \geq 1$ for every $e \in E_n$.

Let $(v_1^n, \dots, v_k^n) \in V_n^k$ be a vector of random vertices, each chosen independently and uniformly over V_n . Note that $w_n(v_1^n), \dots, w_n(v_k^n)$ are nonnegative, independent and identically distributed random variables with mean

$$\mathbb{E}(w_n(v_i^n)) = \frac{1}{|V_n|} \sum_{v \in V_n} w_n(v) = \frac{1}{n} xn = x.$$

Observe also that

$$(3.1) \quad \mathbb{P}(\{v_1^n, \dots, v_k^n\} \in E_n) = \frac{k!|E_n|}{n^k}.$$

On the other hand, since w_n is a vertex cover of H_n , for $\{v_1^n, \dots, v_k^n\} \in E_n$ we have $\sum_{i=1}^k w_n(v_i^n) \geq 1$ and thus

$$(3.2) \quad \mathbb{P}(\{v_1^n, \dots, v_k^n\} \in E_n) \leq \mathbb{P}\left(\sum_{i=1}^k w_n(v_i^n) \geq 1\right).$$

From (3.1), (3.2) and the assumption that Conjecture 1.1 is true, we conclude that

$$\limsup_{n \rightarrow \infty} \frac{|E_n|}{\binom{|V_n|}{k}} \leq \limsup_{n \rightarrow \infty} \mathbb{P}\left(\sum_{i=1}^k w_n(v_i^n) \geq 1\right) \leq \max\{1 - (1-x)^k, (kx)^k\}.$$

It remains to prove the reverse implication. Let us assume that Conjecture 2.2 is valid for some k and $x \in [0, 1/k]$. Due to Lemma 3.1 it is enough to show that Conjecture 1.1 holds for X_i 's attaining a finite set of values $a_1, \dots, a_m \in [0, 1]$ such that

$$\mathbb{P}(X_i = a_j) = p_j/q_j, \quad j = 1, \dots, m$$

for some positive integers p_j and q_j . Moreover, let r be the smallest common multiple of the numbers $\{q_1, \dots, q_m\}$, and define integers

$$(3.3) \quad p'_j = rp_j/q_j, \quad j = 1, \dots, m.$$

In order to apply Conjecture 2.2, we define hypergraphs with bounded fractional matching number. For $n = 1, 2, \dots$, let $V_n = [nr]$. Observing that $np'_1 + \dots + np'_m = nr$, define a function $w_n : V_n \rightarrow [0, 1]$ in such a way that for each $j = 1, \dots, m$ function $w_n(v)$ takes value a_j precisely np'_j times. Let $H_n = (V_n, E_n)$ be a hypergraph with the edge set

$$E_n = \left\{ e \in \binom{V_n}{k} : \sum_{v \in e} w_n(v) \geq 1 \right\}.$$

In view of (3.3), we have that w_n is a fractional vertex cover of H_n of size

$$\sum_{v=1}^{nr} w_n(v) = \sum_{j=1}^m a_j np'_j = n \sum_{j=1}^m r a_j \frac{p_j}{q_j} = nr \mathbb{E}(X_i) \leq xnr.$$

Hence by (2.3) we have $\nu^*(H_n) = \tau^*(H_n) \leq xnr$ and therefore (2.2) gives

$$(3.4) \quad \limsup_{n \rightarrow \infty} \frac{|E_n|}{\binom{nr}{k}} \leq \max \{1 - (1 - x)^k, (kx)^k\}.$$

Let $(v_1^n, \dots, v_k^n) \in V_n^k$ be a vector of random vertices, each chosen independently and uniformly over V_n . Note that for every n the random variable $w_n(v_i^n)$ has the same distribution as X_i , since, by (3.3),

$$\mathbb{P}(w_n(v_i^n) = a_j) = \frac{np'_j}{|V_n|} = \frac{nrp_j/q_j}{nr} = \frac{p_j}{q_j}, \quad j = 1, \dots, m.$$

Let N_n denote the number of k -element vectors $(v_1, \dots, v_k) \in V_n^k$ of vertices with at least two equal coordinates. We have

$$\begin{aligned} \mathbb{P}(X_1 + \dots + X_k \geq 1) &= \mathbb{P}(w_n(v_1^n) + \dots + w_n(v_k^n) \geq 1) \\ &= \frac{|\{(v_1, \dots, v_k) \in V_n^k : \sum_i w(v_i) \geq 1\}|}{(nr)^k} \\ &= \frac{k!|E_n| + N_n}{(nr)^k} \leq \frac{|E_n|}{\binom{nr}{k}} + \frac{\binom{k}{2}(nr)^{k-1}}{(nr)^k}. \end{aligned}$$

Taking the limit over $n \rightarrow \infty$ and using (3.4) we get that

$$\mathbb{P}(X_1 + \dots + X_k \geq 1) \leq \max\{1 - (1 - x)^k, (kx)^k\}. \quad \square$$

Now Theorem 1.3 follows from Lemma 3.2 and Remark 2.3.

REFERENCES

1. N. Alon, P. Frankl, H. Huang, V. Rödl, A. Ruciński, B. Sudakov, *Large matchings in uniform hypergraphs and the conjectures of Erdős and Samuels*, J. Combin. Th., Ser. A, **119** (2012), 1200–1215.
2. P. Erdős, *A problem on independent r -tuples*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math., **8** (1965), 93–95.
3. P. Erdős and T. Gallai, *On maximal paths and circuits of graphs*, Acta Math. Acad. Sci. Hungar., **10** (1959), 337–356.
4. P. Frankl, *On the maximum number of edges in a hypergraph with a given matching number*, arXiv:1205.6847.
5. P. Frankl, *Improved bounds for Erdős’ Matching Conjecture*, J. Combin. Th., Ser. A, **120** (2013), 1068–1072.
6. W. Hoeffding and S.S. Shrikhande, *Bounds for the distribution function of a sum of independent, identically distributed random variables*, Ann. Math. Statist., **26** (1955), 439–449.
7. T. Luczak, K. Mieczkowska, *On Erdős’ extremal problem on matchings in hypergraphs*, J. Combin. Th., Ser. A, **124** (2014), 178–194.
8. S. M. Samuels, *On a Chebyshev-type inequality for sums of independent random variables*, Ann. Math. Statist., **37** (1966), 248–259.
9. S. M. Samuels, *More on a Chebyshev-type inequality for sums of independent random variables*, Purdue Stat. Dept. Mimeo. Ser., **155** (1968).
10. S. M. Samuels, *The Markov inequality for sums of independent random variables*, Ann. Math. Statist., **40** (1969), 1980–1984.

ADAM MICKIEWICZ UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UL. UMULTOWSKA 87, 61-614 POZNAŃ, POLAND
E-mail address: tomasz@amu.edu.pl

ADAM MICKIEWICZ UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UL. UMULTOWSKA 87, 61-614 POZNAŃ, POLAND
E-mail address: kaska@amu.edu.pl

UNIVERSITY OF OXFORD, MATHEMATICAL INSTITUTE, WOODSTOCK ROAD, OXFORD OX2 6GG, UNITED KINGDOM
E-mail address: matas.sileikis@gmail.com